ASSOUAD-NAGATA DIMENSION OF TREE-GRADED SPACES

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ABSTRACT. Given a metric space X of finite asymptotic dimension $\mathrm{asdim} X \leq n$, we consider a quasi-isometric invariant of the space called dimension function. The space is said to have asymptotic Assouad-Nagata dimension $\mathrm{asdim}_{AN} X \leq n$ if there is a linear dimension function in this dimension. We prove that if X is a tree-graded space (as introduced by C. Drutu and M. Sapir) and for some positive integer n a function f serves as an n-dimensional dimension function for all pieces of X, then the function $300 \cdot f$ serves as an n-dimensional dimension function for X. As a corollary we find a formula for the asymptotic Assouad-Nagata dimension of the free product of finitely generated infinite groups:
$$\mathrm{asdim}_{AN}(G * H) = \max\{\mathrm{asdim}_{AN}(G), \mathrm{asdim}_{AN}(H)\}.$$

1. INTRODUCTION AND PRELIMINARIES

Asymptotic dimension was introduced by Gromov in [10] as a large scale invariant of a metric space. Any finitely generated group can be equipped with a word metric. The idea of Gromov was that asymptotic dimension is an invariant of the finitely generated group, i.e. does not depend on the word metric. The linear version of asymptotic dimension was called asymptotic dimension of linear type [12], asymptotic dimension with Higson property [5] and more recently asymptotic Assouad-Nagata dimension.

Spaces of finite asymptotic Assouad-Nagata dimension have some extra properties that spaces of finite asymptotic dimension do not necessarily have. For example if a metric space is of finite asymptotic Assouad-Nagata dimension then it satisfies nice Lipschitz extension properties (see [11] and [3]). It was proved in [8] that the asymptotic Assouad-Nagata dimension bounds the topological dimension of every asymptotic cone of a metric space. As a consequence of the results of Gal in [9] every metric space of finite asymptotic Assouad-Nagata dimension has Hilbert space compression one.

The list of finitely generated groups of finite asymptotic Assouad-Nagata dimension is somewhat limited. The classes of groups known to be of finite asymptotic Assouad-Nagata dimension are Coxeter groups, abelian groups,

hyperbolic groups, free groups, and some types of Baumslag-Solitar groups. It is unknown if nilpotent groups are of finite asymptotic Assouad-Nagata dimension.

A natural problem is to study the behavior of asymptotic Assouad-Nagata dimension under such operations on groups as free product, HNN-extension, and free product with amalgamation. This problem was solved in [1] in case of asymptotic dimension. A formula for the asymptotic dimension of the free product of two groups was introduced in [2] by Bell, Dranishnikov and Keesling. Unfortunately the techniques used in that paper cannot be extended directly to asymptotic Assouad-Nagata dimension.

The goal of the present paper is to obtain a formula for the asymptotic Assouad-Nagata dimension of the free product of two groups. This result will answer [4, Question 8.8]. The techniques used in our proof allow us to extend the results to a larger class of spaces, the so called Tree-graded spaces.

1.1. **Dimension.** Let (X, d) be a metric space and let r be a positive real number. An *r-scale chain* (or *r*-chain) between two points x and y is defined as a finite sequence of points $\{x = x_0, x_1, ..., x_m = y\}$ such that $d(x_i, x_{i+1}) < r$ for every $i = 0, ..., m - 1$. A subset S of a metric space (X, d) is said to be *r-scale connected* if every two elements of S can be connected by an *r*-scale chain contained in S.

Let $\mathcal{U}$ be a subset of a metric space (X, d). A maximal *r*-scale connected subset of $\mathcal{U}$ is called an *r-scale connected component* of $\mathcal{U}$. Notice that any set $\mathcal{U}$ is a union of its *r*-scale connected components and the distance between any two of the components is at least r.

Definition 1.1. Let s and M be two positive numbers. A metric space (X, d) is said to have *dimension $\leq n$ at the scale s of magnitude M* if there is a cover $\mathcal{U} = \{\mathcal{U}_0, ..., \mathcal{U}_n\}$ of X such that every *s*-scale connected component of each $\mathcal{U}_i$ is *M*-bounded (i.e. the diameter of every component is bounded by M).

An increasing function $f_X : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an *n-dimensional control function* of (X, d) if for any $s > 0$ the space (X, d) has dimension $\leq n$ at the scale s of magnitude $f_X(s)$.

A metric space (X, d) is said to be of *asymptotic dimension* at most n (notation: $\mathrm{asdim}(X, d) \leq n$) if there exists an n-dimensional control function for (X, d).

Definition 1.2. A metric space (X, d) is said to be of *asymptotic Assouad-Nagata dimension* at most n (notation: $\mathrm{asdim}_{AN}(X, d) \leq n$) if there exists a linear n-dimensional control function i.e. there exists two positive constants C and b such that the function $f(x) = C \cdot x + b$ is an n-dimensional control function for (X, d).

1.2. **Tree-graded spaces.** Tree-graded spaces were introduced by C. Drutu and M. Sapir [7] in order to study relatively hyperbolic groups. The main

interest in the notion of tree-graded space resides in the following characterization of relatively hyperbolic groups proved by C. Drutu, D. Osin, and M. Sapir: A finitely generated group G is relatively hyperbolic with respect to finitely generated subgroups $H_1, \ldots, H_n$ if and only if every asymptotic cone $Con^\omega(G; e, d)$ is tree-graded with respect to ω-limits of sequences of cosets of the subgroups H_i [7].

Tree-graded spaces are generalizations of $\mathbb{R}$-trees and have many properties similar to the properties of $\mathbb{R}$-trees. They appear naturally as asymptotic cones of groups. A theory of actions of groups on tree-graded spaces is developed recently by C. Drutu and M. Sapir [6].

Definition 1.3. Let X be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic subsets (called *pieces*) covering X. The space X is called *tree-graded with respect to $\mathcal{P}$* if the following two properties are satisfied:

(T_1) Every two different pieces have at most one common point.
(T_2'') Every simple loop in X is contained in one piece.

The main result of the paper is the following theorem.

Theorem 1.4. *Let X be a geodesic tree-graded space. Let n be a positive integer and r and $f(r) \geq r$ be positive real numbers. If each piece of X has dimension $\leq n$ at the scale r of magnitude $f(r)$, then the space X has dimension $\leq n$ at the scale r of magnitude $300 \cdot f(r)$.*

A simple example of a tree-graded space is a Cayley graph of a free product $G * H$ of finitely generated groups when the generating set for $G * H$ is taken as the union of generating sets for G and H.

Corollary 1.5. *Let G and H be finitely generated infinite groups. Then*
$$asdim_{AN}(G * H) = \max\{asdim_{AN}(G), asdim_{AN}(H)\}.$$

Notice that if two groups G and H are finite, the free product $G * H$ is quasi-isometric to a tree, thus has asymptotic Assouad-Nagata dimension 1. If only one of the groups is finite, say G, then we can reduce the computation of the asymptotic Assouad-Nagata dimension to the infinite case as follows: $asdim_{AN}(H) \leq asdim_{AN}(G * H) \leq asdim_{AN}((G \times \mathbb{Z}) * H) \leq asdim_{AN}(H)$. So, the only interesting case is when both groups are infinite.

1.3. **Idea of the proof of the main theorem.** Let X be a geodesic metric space which is tree-graded with respect to a collection $\mathcal{P} = \{P_\lambda\}_{\lambda \in \Lambda}$ of pieces. Fix a base point $\widehat{x} \in X$. For every piece P we let the projection of $\widehat{x}$ to P be the base point of P (call it x_P). Since each piece P has dimension $\leq n$ at the scale r of magnitude $f(r)$, we fix a coloring $c_P \colon P \to \{0, 1, 2, \ldots, n\}$ of P using $n + 1$ colors such that the base point x_P has color 0.

Let us try now to produce a coloring $c_X \colon X \to \{0, 1, 2, \ldots, n\}$ of X. Put $c_X(\widehat{x}) = 0$. Given any point $x \in X$ consider a geodesic γ from $\widehat{x}$ to x. As this geodesic travels from $\widehat{x}$ to x it passes through different pieces of X by

entering a piece P at the base point x_P and exiting P at (possibly) different point $\pi_P(x)$ which is the projection of the point x to the piece P. The first idea is to add up all the changes of colors along the way γ (modulo $n + 1$):

$$c_X(x) = \sum_P c_P(\pi_P(x)) \pmod{n + 1}$$

Notice that this sum becomes finite if we require each coloring c_P to be zero on the ball $B(x_P, 2r)$. The problem with this coloring c_X is that the path γ may never change color if it changes pieces very often (say stays in one piece for no longer than r). Now we improve the formula by introducing additional changes of color: if the path γ does not change color for longer than $99 \cdot f(r)$, we make it change the color. Namely, let $\beta_1, \ldots, \beta_k$ be (maximal) subintervals of the path γ such that each β_i has its interior c_X-colored in one color and the length $|\beta_i|$ of β_i is at least $99 \cdot f(r)$.

$$c_X'(x) = \sum_P c_P(\pi_P(x)) + \sum_{i=1}^k \left\lfloor \frac{|\beta_i|}{99 \cdot f(r)} \right\rfloor \pmod{n + 1}$$

The reason this coloring may not work is that its restriction to a piece P of the space X may differ from the coloring c_P. It may happen because the second sum in the formula for c_X' has nothing to do with the tree-graded structure of X. Our last modification of the formula for the coloring uses the following definition.

Definition 1.6. Let β be any directed geodesic in a tree-graded space X from a point β_- to a point β_+. Consider the induced tree-graded structure on β. Let $[\beta', \beta_+]$ be the piece of this structure containing β_+. We denote by $\lfloor \beta \rfloor$ the subpath of β from β_- to β'.

Here is the coloring proving that the space X has dimension $\leq n$ at the scale r of magnitude $100 \cdot f(r)$.

$$c_X''(x) = \sum_P c_P(\pi_P(x)) + \sum_{i=1}^k \left\lfloor \frac{|\lfloor \beta_i \rfloor|}{99 \cdot f(r)} \right\rfloor \pmod{n + 1}$$

We will give a slightly different and direct definition of a coloring of X in Section 3 and will prove the theorem there.

2. Tree-graded spaces

We collect in this section all the properties of tree-graded spaces that we use in the proof of our main theorem.

Lemma 2.1 ([7, Proposition 2.17]). *Condition (T_2'') in the definition of tree-graded spaces can be replaced by the following condition:*

(T_2') *For every topological arc $C \colon [0, d] \to X$ and $t \in [0, d]$, let $C[t-a, t+b]$ be a maximal sub-arc of C containing $C(t)$ and contained in one piece. Then every other topological arc with the same endpoints as C must contain the points $C(t - a)$ and $C(t + b)$.*

If X is tree-graded with respect to $\mathcal{P}$ then we can always add some or all one-point subsets of X to $\mathcal{P}$, and X will be tree-graded with respect to a bigger set of pieces. To avoid using extra pieces, we shall assume that pieces cannot contain other pieces.

Given a geodesic γ in X and a piece P of X, we denote by γ_P the intersection of γ with P. By [7, Corollary 2.10], γ_P is either empty or a point or a closed subpath of γ. Notice that γ_P is a piece of the tree-graded structure induced on γ by the structure on X.

Lemma 2.2 ([7, Lemma 2.6]). *Let X be a tree-graded space and P be a piece of X. For every point $x \in X$ there exists a unique point $y \in P$ such that $d(x, P) = d(x, y)$. Moreover, every geodesic joining x with a point of P contains y.*

Definition 2.3. The point y in 2.2 is called *the projection of x onto the piece P* and is denoted $\pi_P(x)$.

We will use the following lemma several times in the following context: if $d(x, y) \leq r$ and $\pi_P(x) \neq \pi_P(y)$, then $d(x, P) \leq r$.

Lemma 2.4 ([7, Lemma 2.8]). *Let X be a tree-graded space and P be a piece of X. If x and y are two points of X with $d(x, y) \leq d(x, P)$, then $\pi_P(x) = \pi_P(y)$.*

Lemma 2.5 ([7, Lemma 2.20]). *Let X be a tree-graded space and P be a piece of X. The projection $\pi_P \colon X \to P$ is a 1-Lipschitz retraction.*

Lemma 2.6. *Let X be a tree-graded space and $\{x_i\}_{i=0}^m$ be an r-chain of points in X. Let ω be a geodesic from x_0 to x_m. If P is a piece of X such that $\omega_P \neq \emptyset$, then ω_P is a path from $\omega_P^- = \pi_P(x_0)$ to $\omega_P^+ = \pi_P(x_m)$ and there is a point x_k of the chain such that $\pi_P(x_k) = \omega_P^-$ and $d(x_k, \omega_P^-) \leq r$. Moreover, any geodesic from x_k to x_m passes through ω_P^-.*

Proof. By 2.1, any geodesic from x_0 to x_m passes through ω_P^-. We consider the maximal index k such that any geodesic from x_k to x_m passes through ω_P^- and $\pi_P(x_k) = \omega_P^-$. If $\pi_P(x_{k+1}) \neq \omega_P^-$, then $d(x_k, P) \leq r$ by 2.4; thus $d(x_k, \omega_P^-) = d(x_k, \pi_P(x_k)) = d(x_k, P) \leq r$. If there is a geodesic from x_{k+1} to x_m not passing through ω_P^-, then by 2.1, any geodesic from x_k to x_{k+1} passes through ω_P^-. Since $d(x_k, x_{k+1}) \leq r$, we have $d(x_k, \omega_P^-) \leq r$. $\qquad\square$

3. Proof of the main theorem

Let us introduce some notations before we define a coloring $c_X^* \colon X \to \{0, 1, 2, \ldots, n\}$ of X. Fix a base point $\widehat{x} \in X$. For every piece P we let the projection of $\widehat{x}$ to P be the base point of P (call it x_P).

Since each piece P has dimension $\leq n$ at the scale r of magnitude $f(r)$, we fix a coloring $c_P' \colon P \to \{0, 1, 2, \ldots, n\}$ of P using $n+1$ colors such that r-components of each color are $f(r)$-bounded. Denote by c_P the coloring of P obtained from c_P' by changing the color of the closed $2r$-neighborhood

$B(x_P, 2r)$ of the base point x_P to the color 0. Notice that r-components of any c_P-color other than 0 are $f(r)$-bounded and r-components of c_P-color 0 are $8 \cdot f(r)$-bounded. Since the component of the point x_P of c_P-color 0 will play an important role in our definition of the coloring c_X^*, we give it a special notation $\mathfrak{C}_P$. Denote by $\mathfrak{C}_P^\circ$ the subset $\mathfrak{C}_P \setminus \{x_P\}$.

Given any point $x \in X$ consider a geodesic γ from $\widehat{x}$ to x. The intersection γ_P of γ with a piece P is either empty or a closed subpath of γ from x_P to $\pi_P(x)$.

Definition 3.1. We call γ_P a *short* piece of γ if its endpoint $\pi_P(x)$ belongs to the set $\mathfrak{C}_P$. Otherwise we call this piece of γ *long*.

For every long piece γ_P we call the *reduced long piece* to the piece $\gamma_P' = \gamma_P \setminus B(x_P, \frac{r}{2})$. Notice that any short piece γ_P has length at most $8 \cdot f(r)$, any long piece has length at least $2r$ and any reduced long piece has length at least $\frac{3 \cdot r}{2}$.

Consider the complement in γ of the interiors of all reduced long pieces of γ. Since there are only finitely many long pieces of γ, this complement is a union of closed subpaths $\{\beta_i\}_{i=1}^k$ of γ. Using Definition 1.6, we define a coloring of X as follows:

$$c_X^*(x) = \sum_P c_P(\pi_P(x)) + \sum_{i=1}^k \left\lfloor \frac{|\lfloor \beta_i \rfloor|}{99 \cdot f(r)} \right\rfloor \quad (\text{mod } n+1)$$

It is easy to check using 2.1 that the color $c_X^*(x)$ does not depend on the choice of the geodesic γ from $\widehat{x}$ to x. The following Lemma claims that (except possibly at the base point x_P) the coloring c_X^* differs from the coloring c_P by a constant modulo $n+1$.

Lemma 3.2. *For any piece P of the tree-graded space X we have either*

$$c_X^*|_{P \setminus x_P} = c_P + c_X^*(x_P) \quad (mod \ n+1)$$

or

$$c_X^*|_{P \setminus x_P} = c_P + c_X^*(x_P) + 1 \quad (mod \ n+1)$$

In particular, in any piece P all r-components of each c_X^*-color are $8 \cdot f(r)$-bounded.

Proof. Let us fix a piece Q of X, a point $x \in Q \setminus \{x_Q\}$, and a geodesic γ from $\widehat{x}$ to x. Clearly $\sum_P c_P(\pi_P(x)) = c_Q(x) + \sum_{P \neq Q} c_P(\pi_P(x))$ and the first sum in the definition of the coloring c_X^* changes by $c_Q(x)$ as we go from x_Q to x. Let $Q-1$ denote the piece that contains x_Q and a small portion of γ just before X_Q. If the piece γ_{Q-1} is long, then $c_X^*(x) = c_Q(x) + c_X^*(x_Q) \quad (\text{mod } n+1)$ because the second sum in the definition of the coloring c_X^* has the same value for the points x and x_Q.

If the piece γ_{Q-1} is short, then the second sum in the definition of the coloring c_X^* may change the value. Since the distance from x_Q to the final endpoint of the β_i included in Q is bounded by $8 \cdot f(r) < 99 \cdot f(r)$, the second sum may change the value by at most one. $\square$

The following Lemma claims that the c_X^*-coloring of r-neighborhood of any piece P (except for r-neighborhood of the set $\mathfrak{C}_P$) is determined by the c_X^*-coloring of the piece P.

Lemma 3.3. *If $d(x, P) \leq r$ and $\pi_P(x) \notin \mathfrak{C}_P$, then $c_X^*(x) = c_X^*(\pi_P(x))$.*

Proof. If γ is a geodesic from $\widehat{x}$ to x, then it passes through both x_P and $\pi_P(x)$ (notice that $x_P \neq \pi_P(x)$). Since the subpath β of γ from $\pi_P(x)$ to x has length $\leq r$, it cannot contain a long piece of γ, therefore the first sum in the definition of the coloring c_X^* gives the same value for $\pi_P(x)$ and for x. Since the piece of γ from x_P to $\pi_P(x)$ is long, the contribution of the subpath β to the second sum in the definition of the coloring c_X^* has value

$$\left\lfloor \frac{||\beta||}{99 \cdot f(r)} \right\rfloor \leq \left\lfloor \frac{r}{99 \cdot f(r)} \right\rfloor = 0. \qquad \square$$

Lemma 3.4. *If an r-chain of points $\{x_i\}_{i=0}^m$ in X of the same c_X^*-color $\mathfrak{c}$ connects two points $x = x_0$ and $y = x_m$ of some piece P of X, then $\{\pi_P(x_i)\}_{i=0}^m$ is an r-chain of points of the same c_X^*-color $\mathfrak{c}$ connecting x and y in P provided none of the points $\pi_P(x_i)$ belongs to $\mathfrak{C}_P$.*

Proof. By 2.5, the projection π_P does not increase distances, so the π_P-image of any r-chain is an r-chain.

We prove $c_X^*(\pi_P(x_k)) = \mathfrak{c}$ by induction on k. Clearly, $\pi_P(x_0) = x_0$ and $c_X^*(\pi_P(x_0)) = c_X^*(x_0) = \mathfrak{c}$. If $d(x_k, P) \leq r$, then $c_X^*(x_k) = c_X^*(\pi_P(x_k))$ by 3.3. If $d(x_k, P) > r$, then $\pi_P(x_k) = \pi_P(x_{k-1})$ by 2.4. Thus $c_X^*(\pi_P(x_k)) = c_X^*(\pi_P(x_{k-1})) = \mathfrak{c}$. $\qquad \square$

Lemma 3.5. *If two points x and y of a piece P have the same c_X^*-color and can be connected by an r-chain of that c_X^*-color in X, then $d(x, y) \leq 36 \cdot f(r)$.*

Proof. Assume that $d(x, y) > 36 \cdot f(r)$. Then at least one of the points x and y (say x) is of distance $> 18 \cdot f(r)$ from the base point x_P of the piece P. Let $\{x_i\}_{i=0}^m$ be an r-chain of points in X of the same c_X^*-color connecting the points $x = x_0$ and $y = x_m$. By 2.5, the chain $\{\pi_P(x_i)\}_{i=0}^m$ is also an r-chain connecting the points x and y in P.

If none of the points $\pi_P(x_i)$ belongs to $\mathfrak{C}_P$, then all points of the chain $\{\pi_P(x_i)\}_{i=0}^m$ have the same c_X^*-color by 3.4. So, the points x and y belong to one r-component of that color in P. Therefore $d(x, y) \leq 8 \cdot f(r)$ by 3.2.

If there is a point x_i with $\pi_P(x_i) \in \mathfrak{C}_P$, we consider the following number

$$k = \max\{l \mid \pi_P(x_i) \notin \mathfrak{C}_P \text{ for all } i \leq l\}$$

Since $d(x_k, x_{k+1}) \leq r$ and $\pi_P(x_k) \neq \pi_P(x_{k+1})$, we have $d(x_k, P) \leq r$ by 2.4. Since $\pi_P(x_k) \notin \mathfrak{C}_P$, we have $c_X^*(x_k) = c_X^*(\pi_P(x_k))$ by 3.3. Then all points of the chain $\{\pi_P(x_i)\}_{i=0}^k$ connecting x to $\pi_P(x_k)$ have the same c_X^*-color by 3.4. Therefore $d(x, \pi_P(x_k)) \leq 8 \cdot f(r)$ by 3.2. Thus $d(x, x_P) \leq d(x, \pi_P(x_k)) + d(\pi_P(x_k), \pi_P(x_{k+1})) + d(\pi_P(x_{k+1}), x_P) \leq 8 \cdot f(r) + r + 8 \cdot f(r) \leq 17 \cdot f(r)$ contradicting our assumption that $d(x, x_P) > 18 \cdot f(r)$. $\qquad \square$

Lemma 3.6. *Let x be a point in X and γ be a geodesic from $\widehat{x}$ to x. Suppose that there is an r-chain $\{x_i\}_{i=0}^m$ from $x = x_0$ to a point $x_m \in \gamma$ such that*

all the points of the chain, except possibly x_m, have the same c_X^*-color $\mathfrak{c}$. Denote by γ' the subpath of γ between x and x_m. Assume that if a long piece γ_P of γ intersects γ', then either $\gamma_P \subset \gamma'$ or $\gamma_P \cap \gamma' = x_m$. Then $d(x, x_m) \leq 140 \cdot f(r)$.

Proof. We consider the complement in γ of the interiors of all long pieces of γ. Since there are only finitely many long pieces of γ, this complement is a union of closed subpaths $\{\beta_i\}_{i=1}^k$ of γ.

Claim 1. Any subpath β' of any intersection $\beta_i \cap \gamma'$ of length $|\beta'| > 10 \cdot f(r)$ contains a point of color $\mathfrak{c}$.

Proof of Claim 1. Denote by y the endpoint of β' closest to x_m. Let γ_P be a short piece of γ containing the point y. The intersection $\gamma_P \cap \beta'$ is a subpath of β' with endpoints y and y'. Since γ_P is short, $d(y, y') \leq 8 \cdot f(r)$. By 2.6, there is an element x_t of the r-chain such that $d(x_t, y') \leq r$ and any geodesic from x_t to x_m passes through y'. The only reason for the point y' to have c_X^*-color different from $c_X^*(x_t) = \mathfrak{c}$ might be due to the change in the second sum of the definition of c_X^*-coloring as we go from y' to x_t. Suppose this change happens. Then the same change will happen if we go from y' along β' for the distance $d(x_t, y') \leq r$ thus arriving at a point on β' of color $\mathfrak{c}$.

Claim 2. If there is a long piece γ_Q contained in γ', then $|\gamma_Q| \leq 17 \cdot f(r)$, it is the only long piece of γ inside γ', and the point x_m is $10 \cdot f(r)$-close to γ_Q.

Proof of Claim 2. Denote by y the point $\pi_Q(x)$. By 2.6, there is an element x_k of the r-chain such that $d(x_k, y) \leq r$ and there is an element x_l of the r-chain such that $d(x_l, x_Q) \leq r$. By 3.3, $c_X^*(y) = c_X^*(x_k) = \mathfrak{c}$. Denote by H the r-chain $y, x_k, x_{k+1}, \ldots, x_{l-1}, x_l, x_Q$. By 2.5, the projection $\pi_Q(H)$ is also an r-chain from y to x_Q. Consider the first point x_s of the chain H such that $\pi_Q(x_s) \in \mathfrak{C}_Q$. By 2.4, $d(x_{s-1}, Q) \leq r$ and by 3.3, $c_X^*(\pi_Q(x_{s-1})) = c_X^*(x_{s-1}) = \mathfrak{c}$. Therefore $y, x_{k+1}, \ldots, x_{s-1}, \pi_Q(x_{s-1})$ is an r-chain of color $\mathfrak{c}$. By 3.4, the chain $y, \pi_Q(x_{k+1}), \ldots, \pi_Q(x_{s-1}), \pi_Q(x_{s-1})$ is also an r-chain of color $\mathfrak{c}$ (thus by 3.2, $d(y, \pi_Q(x_{s-1})) \leq 8 \cdot f(r)$). If the set $\mathfrak{C}_Q^\circ$ had c_X^*-color $\mathfrak{c}$, the chain $y, \pi_Q(x_{k+1}), \ldots, \pi_Q(x_{s-1}), \pi_Q(x_s)$ would show that the point y belongs to $\mathfrak{C}_Q$ contradicting the definition of long piece. So, $c_X^*(\mathfrak{C}_Q^\circ) \neq \mathfrak{c}$. As one goes along a geodesic from the point $\pi_Q(x_s)$ to the point x_s, the c_X^*-color changes from $c_X^*(\pi_Q(x_s)) = c_X^*(\mathfrak{C}_Q^\circ)$ to $c_X^*(x_s) = \mathfrak{c}$. The only reason for this change is due to the change in the second sum of the definition of c_X^*-coloring. Therefore $c_X^*(\mathfrak{C}_Q) = \mathfrak{c} - 1 \pmod{n+1}$ and the long piece γ_Q is preceded by some β_j of length at least $99 \cdot f(r) - d(x_s, x_Q) \geq 90 \cdot f(r)$. This would contradict Claim 1 unless $d(x_m, x_Q) < 10 \cdot f(r)$. Now we estimate $|\gamma_Q| = d(y, x_Q) \leq d(y, \pi_Q(x_{s-1})) + d(\pi_Q(x_{s-1}), \pi_Q(x_s)) + d(\pi_Q(x_s), x_Q) \leq 8 \cdot f(r) + r + \mathrm{diam}(\mathfrak{C}_Q) \leq 17 \cdot f(r)$.

Now we finish the proof of lemma. If γ' does not contain a long piece, it is contained in some β_i and by Claim 1 it cannot contain subpaths of color other than $\mathfrak{c}$ of length more than $10 \cdot f(r)$. By definition of the coloring c_X^*,

the color of β_i changes with periodicity $99 \cdot f(r)$. Thus γ' cannot be longer than $(10 + 99 + 10) \cdot f(r)$.

If γ' contains a long piece γ_Q, then $d(x, x_m) = d(x, \pi_Q(x)) + |\gamma_Q| + d(x_Q, x_m)$. By Claim 2, the subpath of γ between the points x and $\pi_Q(x)$ does not contain a long piece and it changes color with periodicity $99 \cdot f(r)$. By Claim 1, this subpath cannot be longer than $(10 + 99) \cdot f(r)$. By Claim 2, $|\gamma_Q| \le 17 \cdot f(r)$ and $d(x_Q, x_m) \le 10 \cdot f(r)$. Finally,

$$d(x, x_m) \le (10 + 99) \cdot f(r) + 17 \cdot f(r) + 10 \cdot f(r) < 140 \cdot f(r).$$

$\square$

The rest of this section is devoted to the proof that r-components in X of each c_X^*-color are $300 \cdot f(r)$-bounded. Consider two points x' and x'' in X. Assume these point have the same c_X^*-color and denote it by $\mathfrak{c}$. Suppose that there exists an r-chain $\{x_i\}_{i=0}^l$ of color $\mathfrak{c}$ from $x' = x_0$ to $x'' = x_l$. Our goal is to show that $d(x', x'') \le 300 \cdot f(r)$.

If the points x' and x'' belong to one piece, then $d(x', x'') \le 36 \cdot f(r)$ by 3.5.

Suppose that the base point $\widehat{x}$ of the space X and one of the points x' or x'' (say x'') belong to one piece Q. Denote by z' the point $\pi_Q(x')$. Consider the following number $k = \max\{l \mid \pi_Q(x_i) = z'$ for all $i \le l\}$. By 2.4, $d(x_k, z') \le r$. By definition of the coloring c_X^*, we have

$$\mathfrak{c} = c_X^*(x_k) = c_Q(\pi_Q(x_k)) + \left\lfloor \frac{||\lfloor\beta\rfloor||}{99 \cdot f(r)} \right\rfloor = c_Q(z') + 0 = c_X^*(z')$$

where β is the part of a geodesic from $\widehat{x}$ to x_k between z' and x_k. Therefore the chain $x' = x_0, x_1, \ldots, x_k, z', x_k, x_{k+1}, \ldots, x_m = x''$ is an r-chain of color $\mathfrak{c}$. By 3.6, $d(x', z') \le 140 \cdot f(r)$. By 3.5, $d(z', x'') \le 36 \cdot f(r)$. So, $d(x', x'') \le d(x', z') + d(z', x'') \le 200 \cdot f(r)$.

Now we assume that no two of the points x', x'', $\widehat{x}$ belong to one piece. Let γ' (resp. γ'') be a geodesic from $\widehat{x}$ to x' (resp. x''). Let ω be a geodesic from x' to x''. Without loss of generality we may assume that any two of these geodesics intersect along a common subpath. Let x' and y' (resp. x'' and y'') be the endpoints of the intersection $\gamma' \cap \omega$ (resp. $\gamma'' \cap \omega$). Let $\widehat{x}$ and $\widehat{y}$ be the endpoints of the intersection $\gamma' \cap \gamma''$.

Suppose $y' = y''$ and the piece of ω containing the point y' is equal to y'. By 2.6, there is an element x_k of the r-chain such that $d(x_k, y') \le r$. Then we have r-chains $x' = x_0, \ldots, x_k, y'$ and $y', x_k, x_{k+1}, \ldots, x_m = x''$. By 3.6, $d(x', y') \le 140 \cdot f(r)$ and $d(y', x'') \le 140 \cdot f(r)$. Thus $d(x', x'') \le 280 \cdot f(r)$.

Suppose $y' \neq y''$. Then there is a simple loop formed by parts of geodesics γ', γ'', and ω between points y', y'', and $\widehat{y}$. By 1.3, this loop belongs to one piece Q. We also consider here the remaining possibility that $y' = y''$ and this point belongs to a non-trivial piece ω_Q of the geodesic ω.

Denote by $z' = \pi_Q(x')$ and $z'' = \pi_Q(x'')$ the endpoints of the piece ω_Q. Then

$$d(x', x'') = d(x', z') + d(z', z'') + d(z'', x'')$$

By 2.6, there is a point x_k of the chain from x' to x'' which is r-close to z'. By 3.6, $d(x', z') \leq 140 \cdot f(r)$. Similarly, $d(z'', x'') \leq 140 \cdot f(r)$.

Now we estimate $d(z', z'')$. By 2.6, there is a point x_k of the chain such that $\pi_Q(x_k) = z'$ and $d(x_k, z') \leq r$. Similarly, there is a point x_l of the chain such that $\pi_Q(x_l) = z''$ and $d(x_l, z'') \leq r$. Thus we have an r-chain $z', x_k, x_{k+1}, \ldots, x_l, z''$. Denote this chain by H. If both points z' and z'' belong to $\mathfrak{C}_Q$, then $d(z', z'') \leq 8 \cdot f(r)$. If the points z' and z'' do not belong to $\mathfrak{C}_Q$, then $c_X^*(z') = c_X^*(x_k) = \mathfrak{c}$ and $c_X^*(z'') = c_X^*(x_l) = \mathfrak{c}$ by 3.3; therefore $d(z', z'') \leq 36 \cdot f(r)$ by 3.5.

It remains to consider the case when one of the points z', z'' (say z'') belongs to $\mathfrak{C}_Q$ and the other point does not belong to $\mathfrak{C}_Q$. By 2.5, the projection $\pi_Q(H)$ is also an r-chain from z' to z''. Consider the first point x_s of the chain H such that $\pi_Q(x_s) \in \mathfrak{C}_Q$. By 2.4, $d(x_{s-1}, Q) \leq r$ and by 3.3, $c_X^*(\pi_Q(x_{s-1})) = c_X^*(x_{s-1}) = \mathfrak{c}$. Therefore $z', x_k, x_{k+1}, \ldots, x_{s-1}, \pi_Q(x_{s-1})$ is an r-chain of color $\mathfrak{c}$. By 3.4, the chain $z', \pi_Q(x_k), \pi_Q(x_{k+1}), \ldots, \pi_Q(x_{s-1}), \pi_Q(x_{s-1})$ is also an r-chain of color $\mathfrak{c}$ (thus by 3.2, $d(z', \pi_Q(x_{s-1})) \leq 8 \cdot f(r)$). Now we estimate $d(z', z'') \leq d(z', \pi_Q(x_{s-1})) + d(\pi_Q(x_{s-1}), \pi_Q(x_s)) + d(\pi_Q(x_s), z'') \leq 8 \cdot f(r) + r + \operatorname{diam}(\mathfrak{C}_Q) \leq 17 \cdot f(r)$.

Finally,

$$d(x', x'') = d(x', z') + d(z', z'') + d(z'', x'') \leq 140 \cdot f(r) + 17 \cdot f(r) + 140 \cdot f(r) < 300 \cdot f(r).$$

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